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OPTIMIZATION OF A CONDUCTING COOLING FIN WITH A HEAT SINK USING OPTIMALITY CRITERION†

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Abstract—An optimality criterion is used to obtain a minimum volume design of a conducting fin with an arbitrarily located heat sink whose volume is a design variable. The finite element method is used for the analysis. The effect of the heat sink as an active cooling device on the optimum volume of the fin is studied.

NOTATION

A area
A_0 reference area
$\bar{A}$ non-dimensional area (A/A_0)
$[C]$ conduction matrix
F Lagrangian
j iteration number
K thermal conductivity
l element length
L length of the fin
n number of elements
N_G heat generation number $\dot{q}L^2/T_0KA_0$
N_s non-dimensional sink strength $\dot{q}_sL^2/T_0K$
$\dot{q}$ heat generated in the fin per unit length
$\dot{q}_s$ heat sink strength
$\{q\}$ thermal load vector
T_C constraint temperature
T_L temperature at $x = L$
$\{T\}$ temperature vector
T_0 reference temperature
$\bar{T}$ non-dimensional temperature T/T_0
V volume
$\bar{V}$ non-dimensional volume V/A_0L
x axial coordinate
λ Lagrangian multiplier

Subscripts
i corresponds to the ith element
j corresponds to the jth iteration
s corresponds to the heat sink

1. INTRODUCTION

Optimality criterion methods are well developed in structural mechanics to search for optimum designs of structures[1–3] subject to a variety of constraints. Recently, an optimality criterion[4] has been developed to obtain optimum design of structures subjected to thermal environment with temperature constraints. This thermal optimality criterion has been successfully tested on various one- and two-dimensional problems. Results obtained from the optimality

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criterion agree within 2% with results from a closed-form solution for one-dimensional problems taken from Ref. [5], and with results from a mathematical programming technique for the two-dimensional problem.

Recent developments in high-temperature structures include the concept of active cooling where part of the heat is removed by heat sinks properly located at some places on the structure. Therefore, in the design of a structure with active cooling devices, it is necessary to include some parameters of the heat sink also as design variables. This paper presents an attempt in this direction. The thermal optimality criterion developed in Ref. [4] is extended to contain a parameter of the heat sink (its volume) as a design variable and is applied to the same problem of obtaining an optimum configuration of the conducting fin of Ref. [5] with a heat sink located at an arbitrary point on the fin.

The solution of the heat conduction problem is based on the classical finite element method [6]. A one-dimensional finite element is used with a cubic temperature distribution along the element. The nodal degrees of freedom of the element are T and $\mathrm{d}T/\mathrm{d}X$. As each element is of uniform cross section, the optimum configuration of the fin obtained from the present method is a stepped one. The volumes of the fin and heat sink along with the area and temperature distributions for optimum total volume of fin-sink combination are presented in the form of tables for two locations of the heat sink and two sink strengths.

2. FORMULATION

Consider a cooling fin maintained at temperature $T = 0$ at $X = 0$, and insulated $((\mathrm{d}T/\mathrm{d}X) = 0)$ at $X = L$, where L is the length of the fin. Heat is generated in the fin at a uniform rate of q per unit length. At a point of the fin (say $X = X_s$), heat sink of strength $\dot{q}_s$ is situated. The problem is to find the volume and area distribution of the fin and volume of the sink for optimum total volume of fin-sink combination such that the temperature at the insulated end is T_C.

Derivation of optimality criterion

The matrix equation governing the heat conduction problem is

$$[C]\{T\} = \{q\} \tag{1}$$

where $[C]$ is the conduction matrix, $\{T\}$ is the temperature vector, and $\{q\}$ is the thermal load vector.

The total volume, V, of the fin and the heat sink is given by

$$V = V_s + \sum_{i=1}^{n} A_i l_i \tag{2}$$

where V_s is the volume of the heat sink, A_i is the area of the ith finite element, l_i is the length of the ith element and n is the number of elements.

The constraint equation is

$$T_L - T_C = 0 \tag{3}$$

or

$$\{T\}^T\{S\} - T_C = 0 \tag{4}$$

where T_L is the temperature of the fin at $X = L$, $\{S\}$ is a vector of zeros except at those degrees of freedom where a constraint is specified and where it has a value of unity, T_C is the constraint temperature and superscript T denotes matrix transposition.

A Lagrangian F, using a Lagrangian multiplier λ, is

$$F = \sum_{i=1}^{n} A_i l_i + V_s + \lambda[\{T\}^T\{S\} - T_c]. \tag{5}$$

Differentiating F with respect to A_i we get

$$\frac{\partial F}{\partial A_i} = l_i + \lambda \frac{\partial \{T\}^T}{\partial A_i} \{S\} = 0. \tag{6}$$

Writing

$$\{S\} = [C]\{r\} \tag{7}$$

where $\{r\}$ is the temperature distribution for the "fictitious" load vector $\{S\}$, eqn (6) becomes

$$l_i + \lambda \frac{\partial \{T\}^T}{\partial A_i} [C]\{r\} = 0. \tag{8}$$

From eqn (1),

$$\frac{\partial \{T\}^T}{\partial A_i} [C] = -\{T\}^T \frac{\partial [C]}{\partial A_i} = -\{T\}^T \frac{[C]_i}{A_i} \tag{9}$$

since the conduction matrix $[C]$ is linearly dependent on the areas A_i.

From eqns (6) and (9), we obtain

$$l_i - \lambda \frac{\{T\}_i^T [C]_i \{r\}_i}{A_i} = 0. \tag{10}$$

The optimality criterion for the area of the elements can be expressed

$$\frac{\lambda \{T\}_i^T [C]_i \{r\}_i}{l_i A_i} = 1. \tag{11}$$

Similarly, differentiating F with respect to V_s, and replacing $\{T\}^T\{S\}$ by $\{q\}^T\{r\}$, the optimality criterion for the volume of the heat sink is obtained as

$$-\frac{\lambda \{q\}^T \{r\}}{V_s} = 1. \tag{12}$$

Note that the implicit assumption is made in eqn (12) that the temperature in the fin is linearly dependent on the sink volume.

The optimality criterion states that if the fin area distribution A_i and sink volume V_s satisfy eqns (11) and (12), the Lagrangian F will be optimum and thus the design for the fin with the heat sink is an optimum one. The values of A_i and V_s are obtained through recurrence relations which are given below. The value of the Lagrangian multiplier is also evaluated using a recurrence relation.

Recurrence relations for design variables

Recurrence relations can be derived from the optimality criteria. Multiplying eqns (11) and (12) by A_i^a and V_s^b and taking the ath root and bth root of both sides, respectively, we get

$$A_i = A_i \left(\lambda \frac{\{T\}_i^T [C]_i \{r\}_i}{l_i A_i} \right)^{1/a} \tag{13}$$

and

$$V_s = V_s \left(-\lambda \frac{\{q\}^T \{r\}}{V_s} \right)^{1/b}. \tag{14}$$

Equations (13) and (14) can be written as the recurrence relations

$$A_{i_{j+1}} = A_{i_j}\left(\lambda \frac{\{T\}_{i_j}^T[C]_{i_j}\{r\}_{i_j}}{l_i A_{i_j}}\right)^{1/a} \tag{15}$$

and

$$V_{s_{j+1}} = V_{s_j}\left(-\lambda \frac{\{q\}_j^T\{r\}_j}{V_{s_j}}\right)^{1/b} \tag{16}$$

where j is the iteration number.

Recurrence relation for λ

To evaluate the value of λ which is to be used in eqns (15) and (16), the constraint equation is used. The constraint equation is

$$T_L = T_C. \tag{17}$$

Multiplying both sides by λ^c, taking the cth root and writing in the recurrence relation form as has been done in the case of the design variables, we get

$$\lambda_{j+1} = \lambda_j\left(\frac{T_L}{T_C}\right)^{1/c}. \tag{18}$$

It has been shown in Ref. [4] that values of $a = 2$ and $c = 2$ in eqns (15) and (18), respectively, give rapid convergence for A_i and λ. Further, it has been found in the present study that $b = 1$ (in eqn 16) gives good convergence for V_s. Hence, in the present study the values for a, b and c are chosen as 2, 1 and 2, respectively.

3. NUMERICAL RESULTS AND DISCUSSION

The optimality criterion has been applied to obtain the optimum volume of a conducting fin with a heat sink (1) located at the midpoint of the fin, and (2) located at the quarter point from the insulated end of the fin. Heat is generated in the fin at a uniform rate ($N_G = 1.0$). Two constraint temperatures and two values of the sink strength are considered in this paper. A five-element (elements of equal length) idealization is used in this paper and the results are presented after twenty iteration cycles.

Table 1 gives the fin volumes and sink volumes for optimum total volume of fin-sink combination for two constraint temperatures and two sink strengths. From this table it is seen that for a given constraint temperature the volume of the fin decreases and the sink volume increases as the sink strength increases. Also, as the sink strength increases, the fin volume decreases more rapidly and the sink volume increases significantly for a lower temperature constraint compared to the case of a higher temperature constraint. This means that if the

Table 1. Cooling fin with a central heat sink—optimum volumes ($\bar{V}$)

Strength of Heat Sink N_s	$\bar{T}_c = 1.0$		$\bar{T}_c = 0.3$	
	Fin Volume	Heat Sink Volume	Fin Volume	Heat Sink Volume
- 0.1	0.4542	0.0201	1.4871	0.2202
- 0.3	0.4478	0.0596	1.2799	0.5770
Without Heat Sink (Ref. [4])	0.4550	--	1.5168	--

Table 2. Cooling fin with a central heat sink—optimum area distribution ($\bar{A}$)

Element Number	$\bar{T}_c = 1.0$			$\bar{T}_c = 0.3$		
	$N_s = -0.1$	$N_s = -0.3$	Without Heat Sink Ref. [4]	$N_s = -0.1$	$N_s = -0.3$	Without Heat Sink Ref. [4]
1	0.5828	0.5736	0.5839	1.9042	1.6125	1.9464
2	0.5342	0.5247	0.5354	1.7409	1.4375	1.7846
3	0.4772	0.4701	0.4781	1.5608	1.3321	1.5938
4	0.4008	0.3971	0.4013	1.3204	1.1966	1.3377
5*	0.2760	0.2734	0.2764	0.9089	0.8207	0.9213

*Element at the insulated end.

Table 3. Cooling fin with a central heat sink—optimum temperature distribution ($\bar{T}$)

Nodal Number	$\bar{T}_c = 1.0$			$\bar{T}_c = 0.3$		
	$N_s = -0.1$	$N_s = -0.3$	Without Heat Sink Ref. [4]	$N_s = -0.1$	$N_s = -0.3$	Without Heat Sink Ref. [4]
1	0.0	0.0	0.0	0.0	0.0	0.0
2	0.3098	0.3092	0.3099	0.0927	0.0906	0.0930
3	0.5714	0.5693	0.5717	0.1706	0.1634	0.1715
4	0.7806	0.7783	0.7809	0.2333	0.2256	0.2343
5	0.9301	0.9293	0.9301	0.2787	0.2764	0.2790
6*	1.0000	1.0000	1.0000	0.3000	0.3000	0.3000

*Insulated end

Table 4. Cooling fin with a heat sink at quarter point from the insulated end—optimum volumes ($\bar{V}$)

Strength of Heat Sink N_s	$\bar{T}_c = 1.0$		$\bar{T}_c = 0.3$	
	Fin Volume	Heat Sink Volume	Fin Volume	Heat Sink Volume
- 0.1	0.4528	0.0328	1.4374	0.3484
- 0.3	0.4356	0.0950	0.9889	0.7532
Without Heat Sink Ref. [4]	0.4550	--	1.5168	--

Table 5. Cooling fin with a heat sink at quarter point from the insulated end—optimum area distribution ($\bar{A}$)

Element Number	$\bar{T}_c = 1.0$			$\bar{T}_c = 0.3$		
	$N_s = -0.1$	$N_s = -0.3$	Without Heat Sink Ref. [4]	$N_s = -0.1$	$N_s = -0.3$	Without Heat Sink Ref. [4]
1	0.5813	0.5611	0.5839	1.8532	1.3324	1.9464
2	0.5328	0.5128	0.5354	1.6924	1.1800	1.7846
3	0.4755	0.4552	0.4781	1.5005	0.9793	1.5938
4	0.3988	0.3797	0.4013	1.2497	0.7398	1.3377
5*	0.2755	0.2690	0.2764	0.8912	0.7129	0.9213

*Element at the insulated end

Table 6. Cooling fin with a heat sink at quarter point from the insulated end—optimum temperature distribution ($\bar{T}$)

Nodal Number	$\bar{T}_c = 1.0$			$\bar{T}_c = 0.3$		
	$N_s = -0.1$	$N_s = -0.3$	Without Heat Sink Ref. [4]	$N_s = -0.1$	$N_s = -0.3$	Without Heat Sink Ref. [4]
1	0.0	0.0	0.0	0.0	0.0	0.0
2	0.3102	0.3124	0.3099	0.0939	0.1019	0.0930
3	0.5720	0.5745	0.5717	0.1726	0.1824	0.1715
4	0.7810	0.7817	0.7809	0.2346	0.2383	0.2343
5	0.9299	0.9279	0.9301	0.2782	0.2715	0.2790
6*	1.0000	1.0000	1.0000	0.3000	0.3000	0.3000

*Insulated end

constraint temperature is higher, the effect of the sink is negligible; however, the sink plays an important role when the constraint temperature is lower.

The results obtained in Ref. [4] for the case of the fin alone (without heat sink) are also presented in Table 1 for the sake of comparison. It may also be pointed out that for a given combination of sink strength and constraint temperature, the optimum total volume of the fin and the heat sink is greater than the optimum volume of the fin without heat sink. However, since the density of heat sink is much lower than that of the fin (actual values are not presented here as this is a theoretical feasibility study), the mass of the fin-sink combination case presented herein will be less than the mass of the fin problem without heat sink presented in Ref. [4]. Tables 2 and 3 give the area distribution and temperature distribution along the fin, respectively. The results of Ref. [4], where the effect of heat sink is not considered in the analysis, are also presented in these tables for the sake of comparison.

The next three tables present similar results of the conducting fin with the heat sink situated at the quarter point from the insulated end where the temperature constraint is prescribed. The general trend of results for this case is similar to the previous one. It can be seen from Table 4 that the effect of heat sink in obtaining the optimum volume is more when the heat sink is situated nearer to the point where the temperature is constrained.

4. CONCLUDING REMARKS

An optimality criterion approach is successfully applied in this paper to obtain the optimum volume of a conducting fin with an active cooling device such as a heat sink, the volume of which is also a design variable. The results obtained indicate the usefulness of an active cooling device to reduce the mass of a thermally-loaded structure with given temperature constraints. The optimality criteria developed in this paper are general and appear to be both useful and effective in feasibility studies for preliminary design of heat coolant systems.

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